Computer Science & Engineering Department
I. I. T. Kharagpur

Foundations of Computing: CS30053
3rd Year : Autumn Semester
Class Test I (**Answers**)

From 1730hr to 1830hr *Date : 26th August, 2003*

1. Give a *bijection* (in closed form) from the set of natural numbers, $\mathbb{N} = \{0, 1, 2, \cdots\}$ to the set of integers divisible by **5**, $\mathbb{Z}_5 = \{\cdots, -10, -5, 0, 5, 10, \cdots\}$.

   Ans. $f : \mathbb{N} \longrightarrow \mathbb{Z}_5$ is defined as follows.

   $$f(n) = \begin{cases} \frac{5n}{2} & \text{if } n \text{ is even,} \\ -\frac{5(n+1)}{2} & \text{if } n \text{ is odd.} \end{cases}$$

   [**2**]

2. Let $f : A \longrightarrow \mathbb{N}$ and $g : \mathbb{N} \times \mathbb{N} \longrightarrow C$ be *bijections*. Show that there is a bijection from A to C. [Do not use *Schröder-Bernstein* theorem.]

   Ans. We know that $h : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ defined as follows is a bijection.

   $$h(k) = (m, n) \textit{ such that } k + 1 = 2^m(2n + 1)$$

   We also know that composition of bijections is a bijection - $(g \circ h \circ f) : A \longrightarrow C$ is a bijection. [**3**]

3. Let $f : A \longrightarrow B$ be an *injection* and $g : B \longrightarrow A$ be a *surjection*. Justify or refute -

   - '$\mathbf{g} \circ \mathbf{f}$ is always a **bijection**.'
     Ans. Let $A = \{a_1, a_2\}$ and $B = \{b_1, b_2, b_3\}$, so that $f(a_1) = b_1$, $f(a_2) = b_3$ and $g(b_1) = a_2$, $g(b_2) = a_1$ and $g(b_3) = a_2$, implies that $g(f(A)) = \{a_2\}$.
   - 'if $g(f(A)) = A$, then $\mathbf{g} \circ \mathbf{f}$ is a **bijection**.'
     Ans. We show that it is not one-to-one. Let $A = \mathbb{N}$ and $B = \mathbb{E}$, $f(n) = 2n$, $g(0) = 0, g(2) = 1, g(4) = g(6) = 2$ and $g(2k) = k - 1, k > 3$. Now $g(f(2)) = 2 = g(f(3))$. [**3**]

4. A *binary relation* R over a set A with n elements ($|A| = n$) is called *reflexive* if $(\mathbf{a}, \mathbf{a}) \in \mathbf{R}$, **for all** $a \in A$. How many reflexive relations are possible on A. Justify your answer.

   Ans. There are n^2 elements in $A \times A$. All the diagonal elements (a, a), $a \in A$ are to be present to make a relation reflexive. Then we have choice out of $n^2 - n$ elements in $2^{n^2 - n}$ ways. [**3**]

5. Show that there is a *bijection* from the *closed interval* $[0, 1]$ to the *closed interval* $[0, 2]$ on the real line. [**3**]

   Ans. Let $f : [0, 1] \longrightarrow [0, 2]$ be $f(x) = 2x$. It is a bijection.

6. Let V be a *denumerable* set of *variable names*. We inductively define the set of *propositional terms* (P) in the following way.

   - '**0**' and '**1**' are in P.
   - Every $v \in V$ (*variable name*) is in P.
   - If p and q are in P, then so are $(p \Rightarrow q)$ and $\neg p$.
   - Nothing else is in P.

   Justify that P is *denumerable*.

   Ans. The set V can be constructed using two symbols $\{v, I\}$, $V = \{v, vI, vII, \cdots\}$. Other symbols are $\{0, 1, (,), \Rightarrow, \neg\}$. Therefore a propositional term is a string over $\{v, I, 0, 1, (,), \Rightarrow, \neg\}$ and the collection is denumerable. [**3**]

7. Consider the alphabet $\Sigma = \{0, 1, *\}$. The set Σ^n is the collection of all strings over Σ of length exactly n i.e. $\Sigma^0 = \{\varepsilon\}$, $\Sigma^1 = \{0, 1, *\}$, $\Sigma^2 = \{\cancel{0}, \cancel{1}, \cancel{*}, 00, 01, 0*, 10, 11, 1*, *0, *1, **\}$, etc. Consider the string

   $$010101010101010101010101010101$$

   We may describe it as $01 * 1111$; which may be interpreted as '**01**' repeated '**1111**' times. The description is shorter than the actual string.

   Give a proof that every string of Σ^n cannot have a **shorter description** using the symbols of Σ.

   Ans. Total number of strings of length less than n over the three element alphabet is $\sum_{i=1}^{n-1} 3^i < 3^n$, the total number of strings of length n. [**3**]