1. $[\mathbf{2+2+3+3+2+3+3+2+2+3=25}]$

Answer with a short justification whether the following statements are *true* or *false*. No credit will be given for writing only *true* or *false*.

(a) A set may be equinumerous to its proper subset.

Ans. The statement is true. If the set is infinite then a proper subset may be equinumerous to the set. As an example - $\mathbb{N} \equiv \mathbb{E}$. The function $f : \mathbb{N} \longrightarrow \mathbb{E}$, $f(n) = 2n$, is a bijection.

(b) No *left-inverse* of a map $f : A \longrightarrow B$ is present if it is not *one-to-one*.

Ans. Let $f : A \longrightarrow B$ is not one-to-one, but $g : B \longrightarrow A$ is a left-inverse of f i.e. there exists a_0 and a_1 such that $f(a_0) = f(a_1)$ and also $g \circ f = 1_A$. But then $g(f(a_0)) = g(f(a_1))$, implies $a_0 = a_1$, a contradiction. Hence the statement is true.

(c) Given the map $f : A \longrightarrow B$, we define a map $F : \mathcal{P}B \longrightarrow \mathcal{P}A$, so that $F(D) = \{c \in A \ : \ f(c) \in D\}$, $D \subseteq B$. The map F is monoton i.e. $C \subseteq D \Rightarrow F(C) \subseteq F(D)$.

Ans. $F(C) = \{c \in A : f(c) \in C\} = \{c \in A : f(c) \in C \subseteq D\} \subseteq \{c \in A : f(c) \in D\} = F(D)$. Hence the statement is true.

(d) The collection of all finite subsets of natural numbers, $\mathbf{2}^{\mathbb{N}}_{fin} = \{A : \ A \subset \mathbb{N} \text{ and } A \text{ is finite}\}$, is not denumerable.

Ans. The statement is false. Let $f : \mathbf{2}^{\mathbb{N}}_{fin} \longrightarrow \mathbb{N}$ be defined as follows.

$$f(A) = \left\{ \begin{array}{ll} 0 & \textbf{if } A = \emptyset, \\ 2^{a_1} + \cdots + 2^{a_k} & \textbf{if } A = \{a_1, \cdots, a_k\} \ . \end{array} \right\}$$

It can be shown to be a bijection.

(e) A denumerable set of variables can be defined inductively using a *finite alphabet*.

Ans. Consider the alphabet $\{x, 0\}$. Variable names are defined inductively as follows.

- **Basis: x is a variable name.**
- **Induction: If v is a variable name then so is $v0$.**
- **Nothing else is a variable name.**

The variable names are $\{x, x0, x00, x000, \cdots\}$.

(f) Fifteen (15) different equivalence relations can be defined on a set of four elements $A = \{a, b, c, d\}$. [Note that an equivalence relation divides the set into nonempty parts.]

Ans. The statement is true. The different partitions are.

$\{\{a\},\{b\},\{c\},\{d\}\}$	$\{\{a,b\},\{c,d\}\}$	$\{\{a,c\},\{b,d\}\}$	$\{\{a,d\},\{b,c\}\}$
$\{\{a\},\{b,c,d\}\}$	$\{\{b\},\{a,c,d\}\}$	$\{\{c\},\{a,b,d\}\}$	$\{\{d\},\{a,b,c\}\}$
$\{\{a\},\{b\},\{c,d\}\}$	$\{\{a\},\{c\},\{b,d\}\}$	$\{\{a\},\{d\},\{b,c\}\}$	$\{\{b\},\{c\},\{a,d\}\}$
$\{\{b\},\{d\},\{a,c\}\}$	$\{\{c\},\{d\},\{a,b\}\}$	$\{\{a,b,c,d\}\}$	

(g) The λ-term $(\lambda xyz.xz(yz))(\lambda ab.a)(\lambda p.pp)(\lambda xy.y)$ can be reduced to '**false**'.

Ans. The statement is true.

$$\begin{aligned}
(\lambda xyz.xz(yz))(\lambda ab.a)(\lambda p.pp)(\lambda xy.y) &\to_\beta (\lambda yz.(\lambda ab.a)z(yz))(\lambda p.pp)(\lambda xy.y) \\
&\to_\beta (\lambda yz.(\lambda b.z)(yz))(\lambda p.pp)(\lambda xy.y) \\
&\to_\beta (\lambda yz.z)(\lambda p.pp)(\lambda xy.y) \\
&\to_\beta (\lambda z.z)(\lambda xy.y) \\
&\to_\beta (\lambda xy.y) \\
&= \textbf{false}
\end{aligned}$$

(h) Reflexive-transitive closure of a binary relation R on a set A is always an equivalence relation.

Ans. The statement is false because the relation may not be symmetric.

(i) Every β-reduction of any λ-term will give a β-normal form.

Ans. The statement is false. The lambda term $(\lambda x.xx)(\lambda x.xx)$ does not have any β-normal form.

(j) The decision problem, "the vertex **d** is reachable from the vertex **s** in a undirected graph G", can be translated to a decision problem of a language.

Ans. A graph G is represented as a 2-tuple of data (V, E). The collection of encodings of (G, s, d) over an alphabet $\{0, 1, *\}$, so that d is reachable from s in G is a language over $\{0, 1, *\}$. Therefore the decision problem of graph is a decision problem of a language.

2. [**5**]

Let the collection of C* programs that calculate functions on the set of natural numbers, $\mathbb{N} \to \mathbb{N}$, be $\mathcal{C}_\mathbb{N}$. Prove by direct use of Cantor's diagonal argument that $\mathcal{C}_\mathbb{N}$ cannot be *equinumerous* to the collection of all functions on natural numbers ($\mathbb{N}^\mathbb{N}$).

Ans. Let each function from $\mathbb{N}$ to itself be computed by a C* program. Let the program P_i computes the function f_i, $i \in \mathbb{N}$. This can be so written as the collection of C* programs and the set of natural numbers are equinumerous. Consider the function $f : \mathbb{N} \longrightarrow \mathbb{N}$ defined as follows.

$$f(n) = \begin{cases} 5 & \textbf{if } f_n(n) \neq 5 \\ 6 & \textbf{if } f_n(n) = 5. \end{cases}$$

The function f cannot be identified with any f_i

$$f(n) = 5 \textbf{ iff } f_n(n) \neq 5, \textbf{ for all } n \in \mathbb{N}.$$

Therefore all functions on $\mathbb{N}$ cannot be computed by a C* program.

3. **[2 + 2 + 1]**

Give an example of a *fixed point combinator* other than the Curry and the Turing combinators. Show that it is a *fixed point combinator*. Justify that there are infinite many of them.

Ans. Let $A = \lambda cury.y(curry)$ **and** $F = AAAA$**. We get**

$$Fu = AAAAu \;=\; (\lambda cury.y(curry))AAAu \;\to_\beta\; u(AAAAu) \;=\; u(Fu).$$

Let $v_1, v_2, \cdots, v_k$ **be variables such that only two of them (except the** k**th one) are same i.e.** $v_i = v_j$, $1 \leq i < j < k$. **We define** $A = \lambda v_1 \cdots v_i \cdots v_{j-1}v_{j+1} \cdots v_k.v_k(v_1 \cdots v_k)$ **and** $F = \overbrace{A \cdots A}^{k-1}$. F **is a fixed point combinator.**

4. **[10]**

Give an inductive definition of '**remainder m n**' ($m\%n$). Justify that *remainder* is λ-definable. Start from the *Barendregt numeral* and define everything that is required to define *remainder*.

Ans. Barendregt Numerals: $0 \equiv \lambda x.x$ **and** $n + 1 \equiv (F, n) = \lambda x.xFn$**.**

(a)

$$\textbf{rem}\ m\ n \;=\; \begin{cases} m & \textbf{if}\ (\textbf{less}\ m\ n), \\ \textbf{rem}\ (\textbf{min}\ m\ n)n & \textbf{else}. \end{cases}$$

Fixed point of : $\lambda f.\lambda mn.(\textbf{less}\ m\ n)\ m\ (f\ (\textbf{min}\ m\ n)\ n)$.

(b)

$$\textbf{less}\ m\ n \;=\; \begin{cases} T & \textbf{if}\ (\textbf{and}\ (\textbf{isZero}\ m)\ (\textbf{not}\ (\textbf{isZero}\ n))), \\ F & \textbf{if}\ (\textbf{isZero}\ n), \\ \textbf{less}\ (\textbf{pred}\ m)\ (\textbf{pred}\ n) & \textbf{else}. \end{cases}$$

The lambda term is easy.

(c)

$$\textbf{min}\ m\ n \;=\; \begin{cases} 0 & \textbf{if isZero}\ m, \\ m & \textbf{if}\ (\textbf{isZero}\ n), \\ \textbf{min}\ (\textbf{pred}\ m)\ (\textbf{pred}\ n) & \textbf{else}. \end{cases}$$

Again a fixed point.

(d) Definitions of and, pred, isZero are already known.