Injection, Surjection and Bijection

Injective or One-to-one Map

A function from a set A to a set B is called an *injection* or an *one-to-one* map, if no two different elements of A are assigned or mapped to the same element of B.

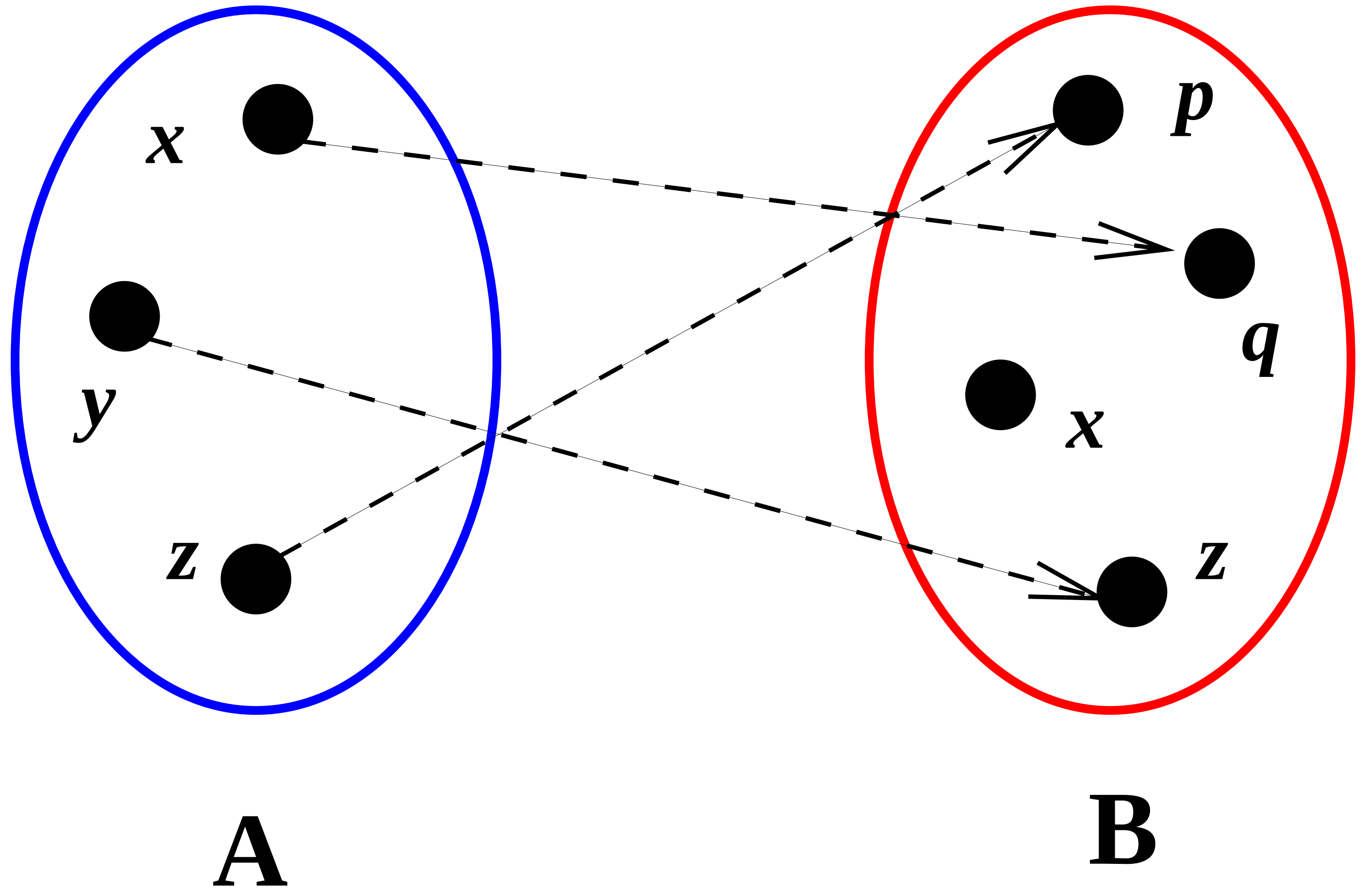

Figure 1: **Injection or One-to-one Map**

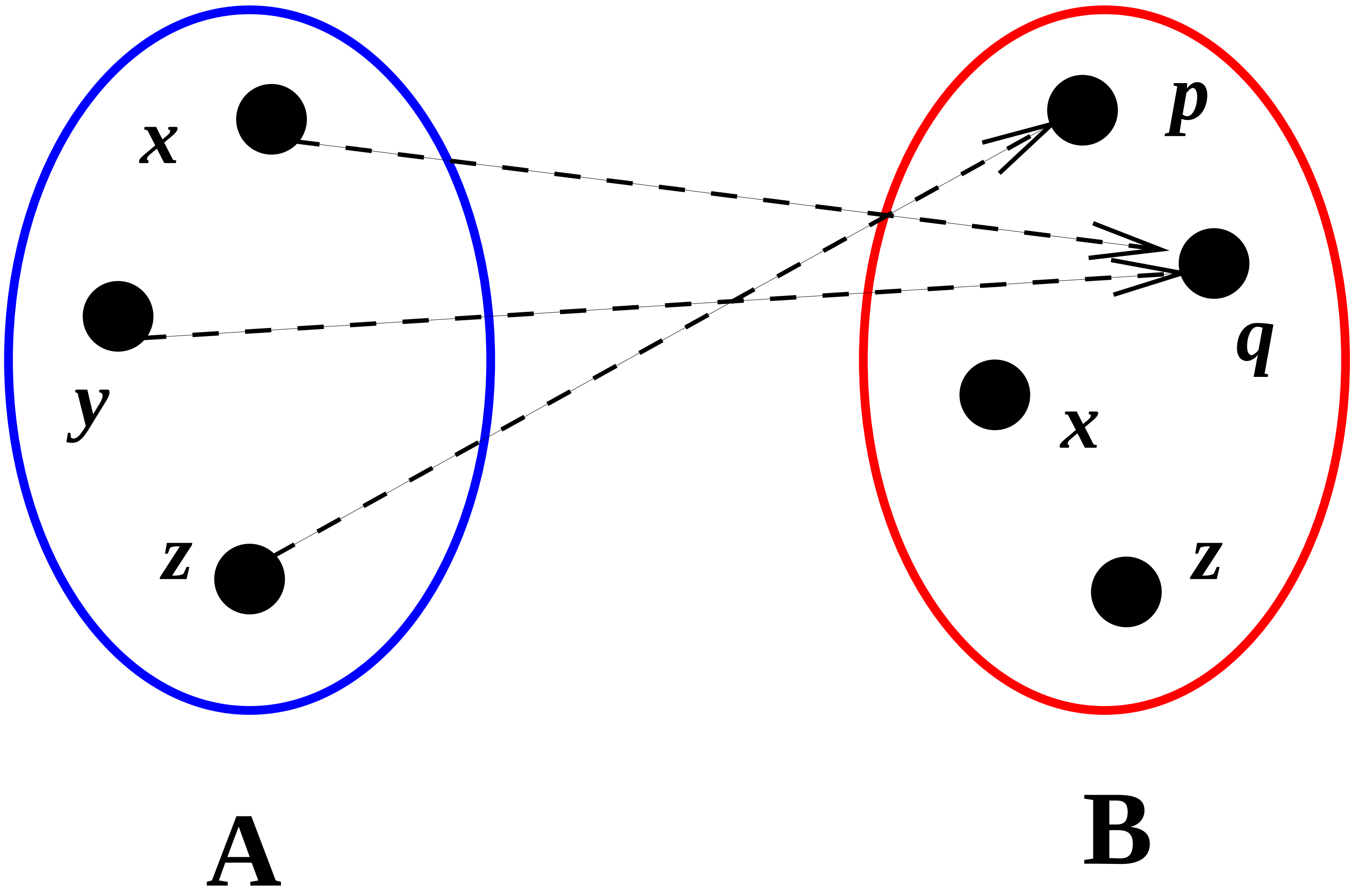

Figure 2: **Not One-to-one**

Formal Definition

A function f from a set A to a set B is called an *injection* or an *one-to-one* map, if

$$f(a) = f(b) \Rightarrow a = b, \text{ for all } a, b \in A.$$

If two *images* are same then their *coimages* are identical. For all $a, b \in A$, $[a \neq b \Rightarrow f(a) \neq f(b)]$ is equivalent to $[f(a) = f(b) \Rightarrow a = b]$ (contrapositive).

Left Inverse

Let $f : A \longrightarrow B$ be a function. A *left inverse* of f, if it exists, is a function $g : B \longrightarrow A$, so that $g \circ f = 1_A$, the identity function on A.

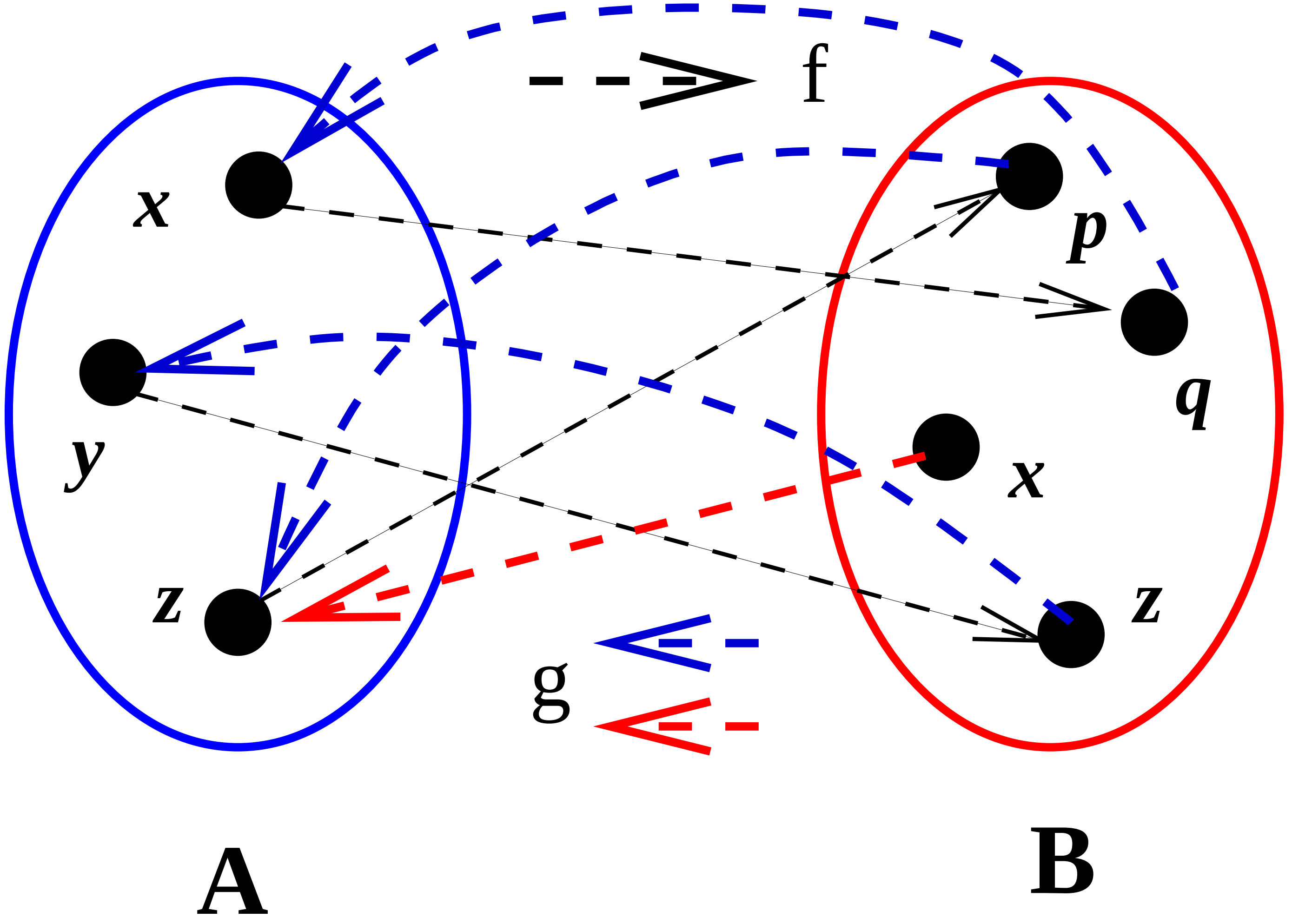

Figure 3: **Left Inverse**

A Little Theorem

Let $f : A \longrightarrow B$ be a map where $A \neq \emptyset$. f is *one-to-one* if and only if f has a *left inverse*.

If A is *null*, there is no function from $B \neq \emptyset$ to A and so the condition.

Proof

The *proof* of this theorem has two parts.

- (*only if* $:\Rightarrow$) : if $f : A \longrightarrow B$ is an *injection*, then f has a *left inverse*.
- (*if* $:\Leftarrow$) : if f has a *left inverse* then f is *one-to-one*.

Proof : $(\Rightarrow)$

We assume the antecedent '$f : A \longrightarrow B$ is an injection' to be *true* (otherwise the statement will be vacuously *true*).

Define $g : B \longrightarrow A$ as follows:

$$g(b) = \begin{cases} a, & \text{if } f(a) = b, \\ a_0, & \text{if } b \text{ is not an image of any } a \in A, \text{ and} \\ & a_0 \text{ is any element of } A. \end{cases}$$

$(g \circ f)(a) = g(f(a)) = g(b) = a$, for all $a \in A$, hence $g \circ f = 1_A$, and g is the *left inverse* of f.

Proof : ($\Leftarrow$)

- We assume that a *left inverse*, say $g : B \longrightarrow A$, of f exists i.e. $g \circ f = 1_A$.
- We further assume that the statement is *false* i.e. f is not *one-to-one*. This implies that A must have at least two elements and there are two distinct elements a_0 and a_1 in A such that $f(a_0) = b = f(a_1)$.
- But then, $a_0 = 1_A(a_0) = (g \circ f)(a_0) = g(f(a_0)) = g(b) = g(f(a_1)) = (g \circ f)(a_1) = 1_A(a_1) = a_1$, is a *contradiction*. Hence the proof by *reductio ad absurdum*.

Another Theorem

A map $f : A \longrightarrow B$ is an *injection* if and only if, for any set X and a pair of maps $g, h : X \longrightarrow A$,

$$f \circ g = f \circ h \;\Rightarrow\; g = h.$$

Proof

Again there are two implications to prove.

- (*only if* :$\Rightarrow$) : if $f : A \longrightarrow B$ is an *injection*, then for any set X and a pair of maps $g, h : X \longrightarrow A$, $f \circ g = f \circ h \;\Rightarrow\; g = h$.
- (*if* :$\Leftarrow$) : if for any set X and a pair of maps $g, h : X \longrightarrow A$, $f \circ g = f \circ h \;\Rightarrow\; g = h$, then $f : A \longrightarrow B$ is an *injection*.

Proof : ($\Rightarrow$)

We assume the antecedent '$f : A \longrightarrow B$ is an injection' to be *true* (otherwise the statement will be vacuously *true*).

The consequence is also in form of an implecation - 'if for any set X and a pair of maps $g, h : X \longrightarrow A$, $f \circ g = f \circ h$, then $g = h$'.

Proof : $(\Rightarrow)$ **(*cont.*)**

We again assume that the antecedent 'if for any set X and a pair of maps $g, h : X \longrightarrow A$, $f \circ g = f \circ h$' is *true*. If X is a *null set* there is nothing to prove as there can be only one map from a *null set* to any other set.

Proof : ($\Rightarrow$) (*cont.*)

For a non-null X,

$$\begin{aligned}
f \circ g = f \circ h \quad &\Rightarrow \quad f_L \circ (f \circ g) = f_L \circ (f \circ h), \\
& \quad f \text{ is an } \textit{injection} \text{ and has a } \textit{left-inverse}. \\
&\Rightarrow \quad (f_L \circ f) \circ g = (f_L \circ f) \circ h, \\
& \quad \text{function composition is associative.} \\
&\Rightarrow \quad 1_A \circ g = 1_A \circ h, \\
& \quad f_L \circ f = 1_A, \text{ identity function.} \\
&\Rightarrow \quad g = h.
\end{aligned}$$

Proof : $(\Leftarrow)$

We give an outline of a *reductio ad absurdum* proof.

- Assume 'if for any set X and a pair of maps $g, h : X \longrightarrow A$, $f \circ g = f \circ h \;\Rightarrow\; g = h$' to be *true*.
- Further assume that f is not an *injection*.
- Construct two functions g, h so that $f \circ g = f \circ h$ but $g \neq h$ - *contradiction*.

Composition of One-to-one Map

If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are *one-to-one*, then so is $g \circ f$.

Proof: Let $a_0, a_1 \in A$, and $(g \circ f)(a_0) = (g \circ f)(a_1)$.

$$
\begin{aligned}
(g \circ f)(a_0) = (g \circ f)(a_1) &\Rightarrow g(f(a_0)) = g(f(a_1)) \\
&\Rightarrow f(a_0) = f(a_1), \\
&\quad\;\; g \text{ is one-to-one} \\
&\Rightarrow a_0 = a_1, \\
&\quad\;\; f \text{ is one-to-one} \\
&\Rightarrow (g \circ f) \text{ is one-to-one.}
\end{aligned}
$$

What Can Be Concluded?

- What can we conclude if there is a *one-to-one* map $f : A \longrightarrow B$?
- Existence of such a map guarantees that for each element a of A, there is a unique element b of B.
- There may be extra elements of B that are not associated to any element of A.
- The '*size*' of B is no less than that of A. [Note that we are not counting the elements of A or B. We are just comparing their relative sizes.]
- A is not larger than B, $A \leq B$.

$\boxed{\textbf{If } f : A \longrightarrow B \textbf{ is One-to-one, } A \leq B}$

The **cardinality** of A is less than or equal to the **cardinality** of B.

Surjection or Onto Map

A function $f : A \longrightarrow B$ is called *onto* B or a *surjection* if $f(A) = B$, where for any $D \subseteq A$,

$$f(D) = \{f(d) : d \in D\} \subseteq B$$

Every element of B is an 'f-*image*' of some $a \in A$.

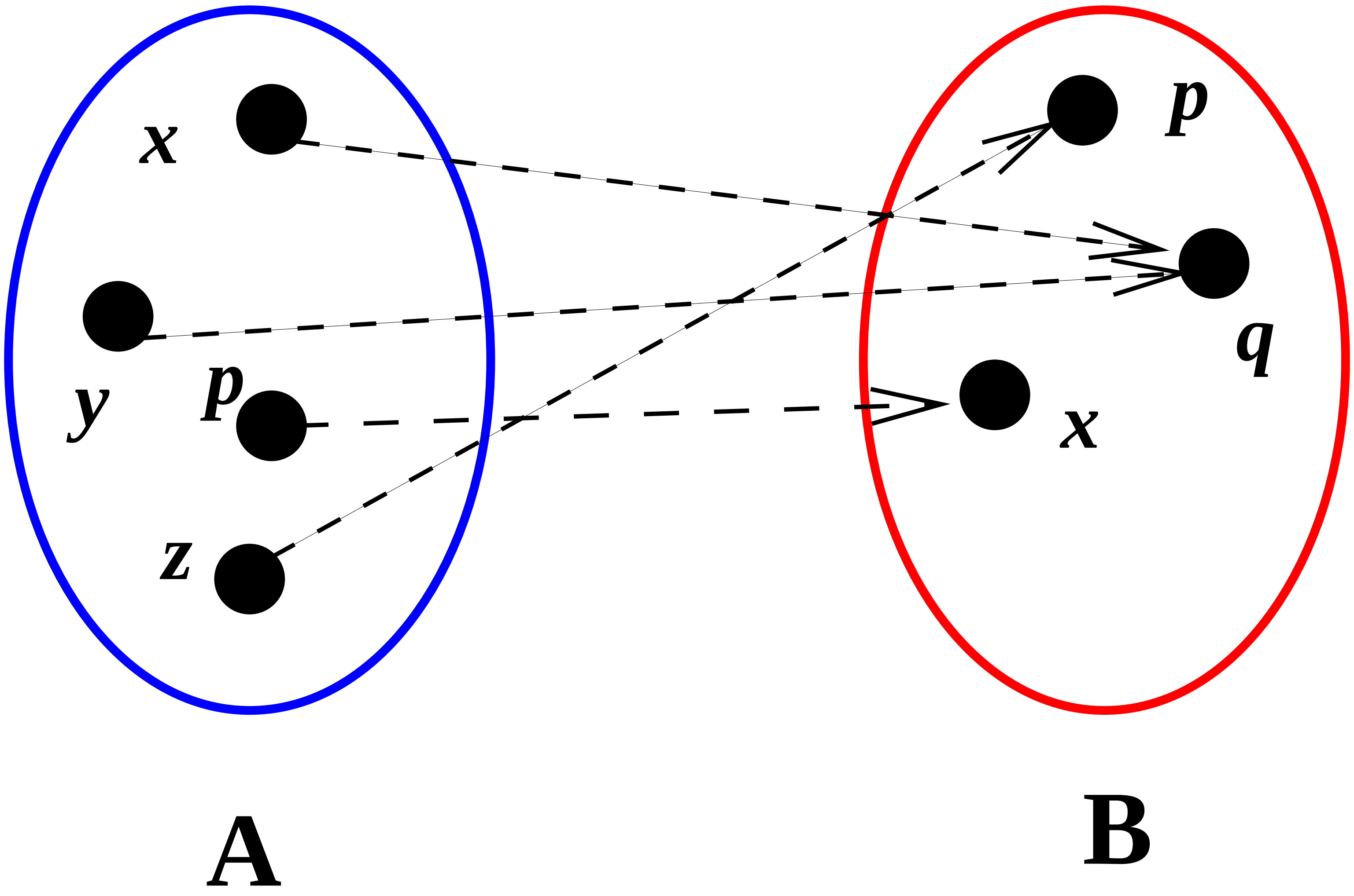

Figure 4: **Surjection or Onto Map**

Right Inverse

Let $f : A \longrightarrow B$ be a map. A *right inverse* of f is a map $g : B \longrightarrow A$, such that $f \circ g = 1_B$, the *identity function* on B.

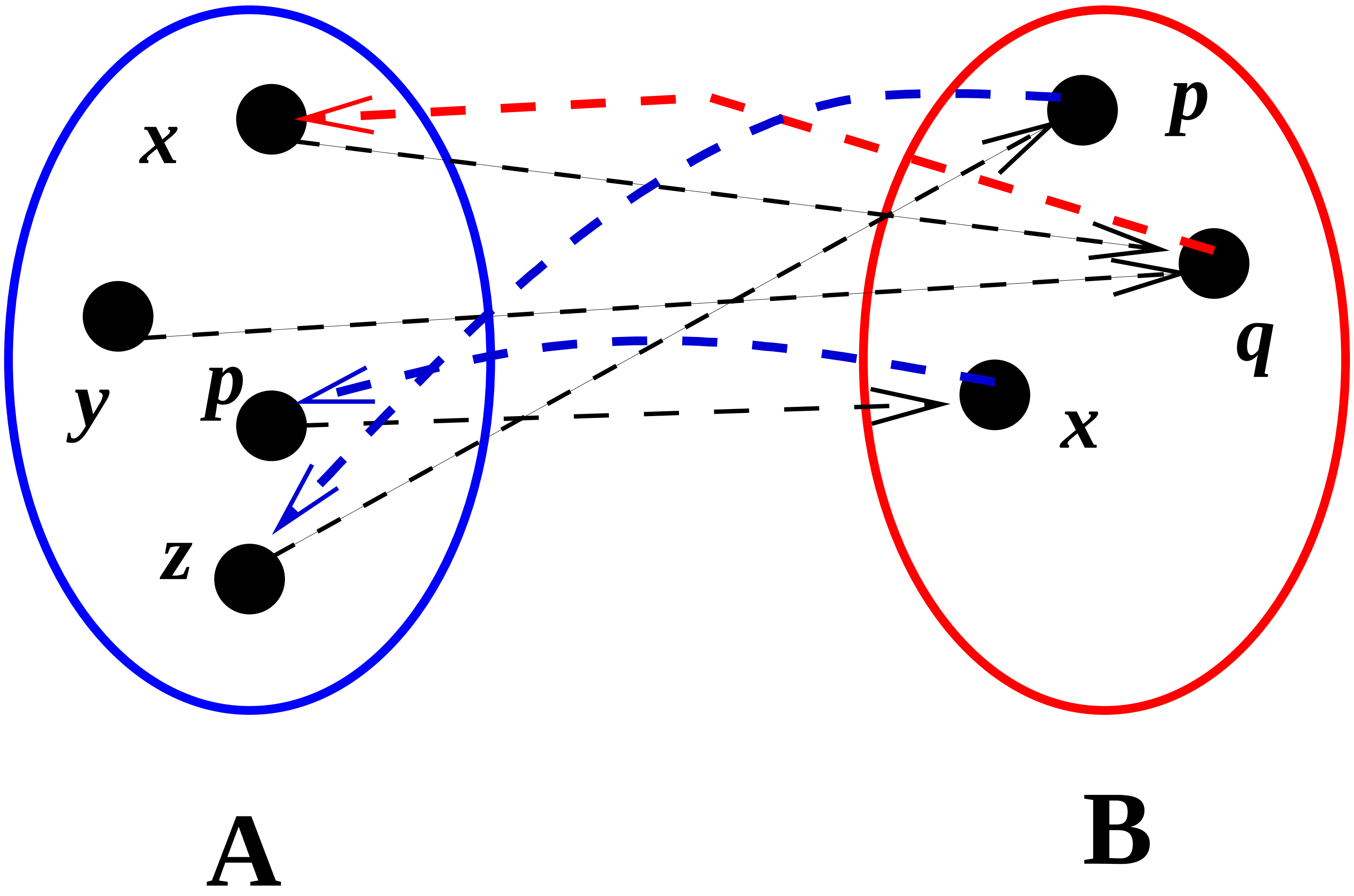

Figure 5: **Right Inverse**

Prove the Following Propositions

- Let $f : A \longrightarrow B$ and $A \neq \emptyset$. f is *onto* B *iff* f has a *right inverse.*
- A map $f : A \longrightarrow B$ is *onto* B if and only if, for any set X and a pair of maps $g, h : B \longrightarrow X$,

$$g \circ f = h \circ f \;\Rightarrow g = h.$$

- If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be *surjections*, then so is $g \circ f$.

What Can Be Concluded?

- What can we conclude if there is an *onto* map $f : A \longrightarrow B$?
- Each element b of B is an *image* of some $a \in A$.
- More than one elements of A may be mapped to an element of B, but no element of B is left without a *coimage*.
- The '*size*' of B is no more than that of A. [We are comparing the relative sizes without counting.]
- B is not larger than A, $B \leq A$.

$\boxed{\textbf{If } f : A \longrightarrow B \textbf{ is Onto, } B \leq A}$

The **cardinality** of B is less than or equal to the **cardinality** of A.

Complementary Concepts (*Dual*)

One-to-one and Onto Maps

Bijection

A function $f : A \longrightarrow B$ is called a *bijection* from A to B if it is both an *injection* (*one-to-one*) as well as a *surjection* (*onto*).

Every element of A is mapped to a unique element of B and each element of B has a *coimage* in A.

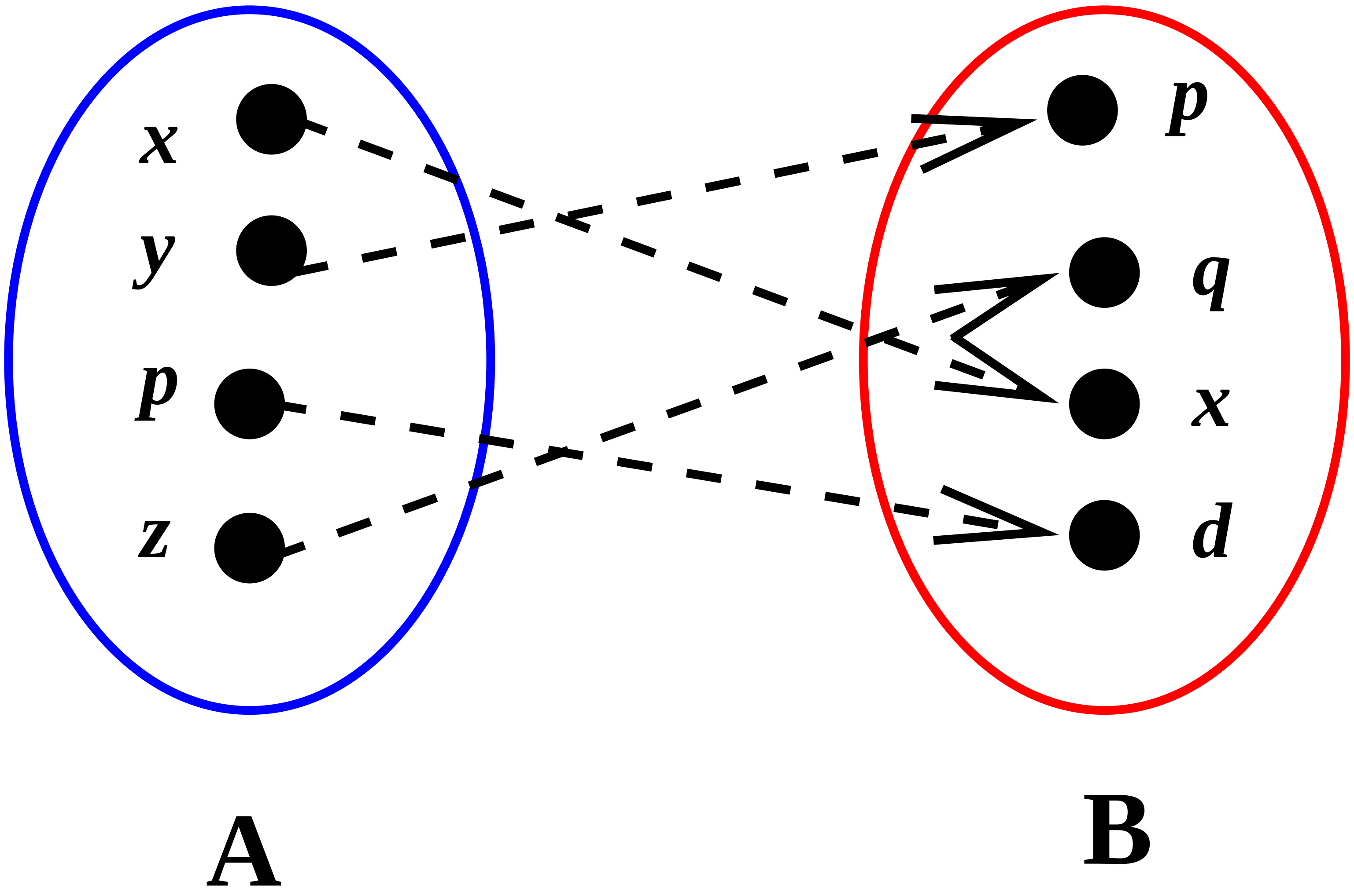

Figure 6: **Bijection**

Unique Inverse of Bijection

Let $f : A \longrightarrow B$ be a *bijection*.

$\Rightarrow$ f is *one-to-one* and *onto*,

$\Rightarrow$ f has a *left-inverse* f_L and

f also has a *right inverse*, f_R.

- $f_L = f_R$,
- Inverse is unique.

Proof

We have $f_L \circ f = 1_A$ and $f \circ f_R = 1_B$.

- $f_L = f_L \circ 1_B = f_L \circ (f \circ f_R) = (f_L \circ f) \circ f_R = 1_A \circ f_R = f_R$.

Proof (*cont.*)

Let $f_L = f_R = f^{-1}$, now we have $f \circ f^{-1} = 1_B$ and $f^{-1} \circ f = 1_A$.

- Let there be another inverse f', then we also have $f \circ f' = 1_B$ and $f' \circ f = 1_A$.
- $f^{-1} = f^{-1} \circ 1_B = f^{-1} \circ (f \circ f') = (f^{-1} \circ f) \circ f' = 1_A \circ f' = f'$.

The *inverse* of a *bijection* is unique.

Composition of Bijection

If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are *bijections* then so is $g \circ f$.

Prove the proposition

What Can Be Concluded?

- What can we conclude if there is a *bijective* map $f : A \longrightarrow B$?
- No two elements of A are mapped to the same element of B and each element of B has a *coimage* in A.
- The '*size*' of A matches with the size of B. [We are not counting.]
- A is *equinumerous* to B, $A \simeq B$.

If $f : A \longrightarrow B$ is a Bijection,

f is an Injection and also a Surjection,

$A \leq B$ and $B \leq A$,

$$\boxed{A \simeq B}$$

The **cardinality** of A is same as the **cardinality** of B.

It is Funny!

A proper subset B of a set A may be *equinumerous* to A.

Consider the set of natural numbers ($\mathbb{N}$) and the set of even numbers ($\mathbb{E}$), $\mathbb{E} \subset \mathbb{N}$.

- Define $f : \mathbb{N} \longrightarrow \mathbb{E}$, so that $f(n) = 2n$.
- It is a bijection with the inverse $f^{-1} : \mathbb{E} \longrightarrow \mathbb{N}$, defined as $f^{-1}(k) = k/2$.
- $(f^{-1} \circ f)(n) = f^{-1}(f(n)) = f^{-1}(2n) = 2n/2 = n$.
- $(f \circ f^{-1})(k) = f(f^{-1}(k)) = f(k/2) = 2 * k/2 = k$.

The Bijection

$$
f : \boxed{\begin{array}{cccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 0 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & \cdots \end{array}} : f^{-1}
$$