School of Mathematical and Computational Sciences

# Indian Association for the Cultivation of Science

*Master's/Integrated Master's-PhD Program/ Integrated Bachelor's-Master's Program/PhD Course*

## Theory of Computation II: COM 5108

*Quiz I (20 August 2025)*

## Answer All Questions

**Marks: 10**

1. Answer with a brief justification (no formal proof is required). **[5]**

   (a) Is the set $\{0,1\}^*$ countably infinite?

   (b) Is the collection of all regular languages over $\{0,1\}^*$ *uncountable*?

   (c) Is the collection of *context-free languages (CFL(* over $\{0,1\}^*$ closed under complementation?

   (d) Is the collection of *decidable/recursive* languages over $\{0,1\}^*$ closed under complementation?

   (e) Is the collection of all *semi-decidable/recursively-enumerable* languages over $\{0,1\}^*$ closed under intersection?

   **Ans.**

   (a) Yes. If we replace 0 by 1 and 1 by 2 the strings are of the following form $\{1,2\}^* = \{\varepsilon, 1, 2, 11, 12, 21, 22, 111, \cdots\}$. There is a natural one-one map from $\{1,2\}^* \to \mathbb{N}_0$, $\varepsilon \mapsto 0$ and $n \neq \varepsilon$ is mapped to $n$.

   (b) No. Any description (DFA or regular expression) of a regular language uses finite number of symbols (say $k$). The description may be viewed as a radix $k+1$ numeral. So there is a one-one map from the collection of descriptions to $\mathbb{N}_0$.

   (c) No. $L = \{x \in \{0,1\}^* : x \neq ww\}$ is a CFL. But $\overline{L} = \{x \in \{0,1\}^* : x = ww\}$ is not.

   (d) Yes. A language $L$ is decidable if there is a Turing machine $M$ that always halts and for any $x \in \{0,1\}^*$, $M(x) = 1$ if $x \in L$; otherwise $M(x) = 0$.
   $\overline{L}$ can be decided by $\overline{M}$ that is same as $M$ except, the outputs are flipped.

(e) Yes. Let $L_1$ and $L_2$ be r.e. languages and $M_1$ and $M_2$ are their Turin machines that recognize them.
If $x \in L_1 \cap L_2$, then both $M_1$ and $M_2$ will halt with $M_1(x) = 1 = M_2(x)$.
We design $M_{L_1 \cap L_2}$ as follows.

i. Emulate $M_1$ on a copy of $x$ (save another copy).
ii. If $M_1(x) = 0$, halt with output 0.
iii. If $M_1(x) = 1$, emulate $M_2$ on the saved copy of $x$.
iv. If $M_2(x) = 1$, output 1;
v. If $M_2(x) = 0$, output 0.

2. Give a bijection from $\mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$. **[2]**

**Ans.** $f((a,b,c)) = 2^d(2c+1-1)$, where $d = 2^a(2b+1)-1$. As an example $(0,1,2) \mapsto 19$ as $d = 2^0(2 \times 1 + 1) - 1 = 2$ and $2^2(2 \times 2 + 1) - 1 = 19$.
*One-one:* Let $f((a,b,c)) = f((p,q,r))$.

$$\begin{aligned}
& 2^{2^a(2b+1)-1}(2c+1) - 1 = 2^{2^p(2q+1)-1}(2r+1) - 1 \\
\Rightarrow\ & 2^{2^a(2b+1)-1}(2c+1) = 2^{2^p(2q+1)-1}(2r+1) \\
\Rightarrow\ & 2^a(2b+1) - 1 = 2^p(2q+1) - 1, 2c+1 = 2r+1 \\
\Rightarrow\ & 2^a(2b+1) = 2^p(2q+1), c = r \\
\Rightarrow\ & a = p, b = q, c = r.
\end{aligned}$$

*Onto:* If $n \in \mathbb{N}_0$ then $n+1 \in \mathbb{N}$, and $n+1 = 2^d(2c+1)$, where $d, c \in \mathbb{N}_0$.
If $d \in \mathbb{N}_0$ then $d+1 \in \mathbb{N}$, and $d+1 = 2^a(2b+1)$, where $a, b \in \mathbb{N}_0$.

3. Prove using *mapping reducibility* that
$L_{CFL} = \{< M >:\ M$ is a Turing machine and $L(M)$ is a CFL$\}$ is undecidable. **[3]**

**Ans.** The $f :< M, x > \mapsto < M_1 >$, where $M_1$ is as follows:
$M_1$ : input $y$

(a) If $y = 0^n1^n0^n$, *accept.*
(b) Simulate $M$ on $x$.
(c) If $M$ halts, *accept.*

$$L(M_1) = \begin{cases} \{0,1\}^* & \text{if } M \text{ halts on } x, \\ \{0^n1^n0^n :\ n \in \mathbb{N}_0\} & \text{if } M \text{ does not halt on } x. \end{cases}$$

Note that $\{0,1\}^*$ is a CFL but $\{0^n1^n0^n :\ n \in \mathbb{N}_0\}$ is not
So $L(M_1)$ is a CFL if and only if $M$ halts on $x$.
If there is a decider for $L_{CFL}$, it can be used to decide $L_H$.